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ON THE LOCATION OF THE ROOTS OF THE JACOBIAN OF TWO  
BINARY FORMS, AND OF THE DERIVATIVE OF A  
RATIONAL FUNCTION\*

BY

J. L. WALSH

The present paper is an extension and in some respects a simplification of a recent paper published under the same title.† Both papers are based on a theorem (Theorem I, below) due to Professor Bôcher.‡ By means of the statical problem of determining the positions of equilibrium in a certain field of force, there are obtained some new results concerning the location of the roots of the jacobian of two binary forms relative to the location of the roots of the ground forms. Application is made to the roots of the derivative of a polynomial and to the roots of the derivative of a rational function. The present paper gives a proof and an application of a geometrical theorem (Theorem II) which may be not uninteresting.

Bôcher considers a number of fixed particles in a plane or by stereographic projection on the surface of a sphere, and supposes each particle to repel with a force equal to its mass (which may be positive or negative) divided by the distance. If the plane is taken as the Gauss plane, the following result is proved:§

**THEOREM I.** *The vanishing of the jacobian of two binary forms  $f_1$  and  $f_2$  of degrees  $p_1$  and  $p_2$  respectively determines the points of equilibrium in the field of force due to  $p_1$  particles of mass  $p_2$  situated at the roots of  $f_1$ , and  $p_2$  particles of mass  $-p_1$  situated at the roots of  $f_2$ .*

The jacobian vanishes not only at the points of no force, but also at the multiple roots of either form or a common root of the two forms; such a point is called a position of *pseudo-equilibrium*.

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\* Presented to the Society, Dec. 31, 1919.

† Walsh, these Transactions, vol. 19 (1918), pp. 291-298. This paper will be referred to as I.

‡ Maxime Bôcher, *A problem in statics and its relation to certain algebraic invariants*, Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), p. 469.

§ Bôcher's proof (l. c., p. 476) is reproduced in I, p. 291.

It is intuitively obvious that there can be no position of equilibrium very near any of the fixed particles, or very near and outside of a circle containing a number of fixed particles, all attracting or all repelling, if the other particles are sufficiently remote. We consider, then, a number of particles in a circle or more generally in a circular region. First we adjoin to the plane the point at infinity, and use the term *circle* to include point and straight line; then we define a *circular region* to be a closed region of the plane bounded by a circle, namely, the interior of a circle, the exterior of a circle including the point at infinity, a half plane, a point, or the entire plane. There will be no confusion in having the same notation for a circular region as for its boundary.

In the following development we shall use several lemmas.

**LEMMA I.** *The force at a point  $P$  due to  $k$  particles each of unit mass situated in a circular region  $C$  not containing  $P$  is equivalent to the force at  $P$  due to  $k$  coincident particles each of unit mass also in  $C$ .*

Denote by  $C'$  the inverse of  $C$  in the circle of unit radius and center  $P$  and by  $Q'$  the inverse of any point  $Q$  with regard to that circle. The force at  $P$  due to a particle at  $Q$  is in direction and magnitude  $PQ'$ . We replace  $k$  vectors  $PQ'$  by  $k$  coincident vectors having one terminal at  $P$  and the other at the center of gravity of the points  $Q'$ ; these two sets of vectors have the same resultant. If any point  $Q$  is in the region  $C$ , its inverse  $Q'$  is in  $C'$ , and the center of gravity of a number of such points  $Q'$  is also in  $C'$ . The inverse of this center of gravity is then in  $C$ .

**LEMMA II.** *In the field of force due to  $k$  positive particles at  $z_1$ ,  $l$  positive particles at  $z_2$ , and  $k + l$  negative particles at  $z_3$ , the only position of equilibrium is  $z_4$  as determined by the cross-ratio*

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \equiv (z_1, z_2, z_3, z_4) = \frac{k + l}{k}.$$

The lemma is evidently true when one of the points  $z_1, z_2, z_3$  is at infinity. The invariance of the positions of equilibrium under linear transformation follows from Theorem I and hence completes the proof.

We shall next prove a preliminary theorem, the proof of which is given in part by several succeeding lemmas.

**THEOREM II.** *If the envelopes of points  $z_1, z_2, z_3$  are circular regions  $C_1, C_2, C_3$  respectively, then the envelope of  $z_4$ , defined by the real constant cross-ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

*is also a circular region.\**

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\* The term *envelope* is used to denote the set of points which is the totality of positions assumed by each of the points  $z_1, z_2, z_3, z_4$ ; the points  $z_1, z_2, z_3$  are supposed to vary independently.

The proof of Theorem II which is presented in detail has some advantages and some dis-

We denote the envelope of  $z_4$  by  $C_4$ , and we must show that  $C_4$  is a region bounded by a single circle. First we consider several special cases of the theorem. If  $C_1$ ,  $C_2$ , and  $C_3$  are distinct points,  $C_4$  is a point. If any of the regions  $C_1$ ,  $C_2$ ,  $C_3$  is the entire plane,  $C_4$  is also the entire plane. If  $\lambda = 0$  and if  $C_1$  and  $C_2$  have a point in common,  $C_4$  is the entire plane. If  $\lambda = 0$  and  $C_1$  and  $C_2$  have no point in common,  $z_3 = z_4$  and so  $C_4$  coincides with  $C_3$ . If  $\lambda = \infty$  and  $C_2$  and  $C_3$  have a common point,  $C_4$  is the entire plane. If  $\lambda = \infty$  and  $C_2$  and  $C_3$  have no common point,  $C_4$  and  $C_1$  are identical. If  $\lambda = 1$  and  $C_1$  and  $C_3$  have a common point,  $C_4$  is the entire plane. If  $\lambda = 1$  yet  $C_1$  and  $C_3$  have no common point,  $C_4$  is identical with  $C_2$ . In the sequel, unless it is explicitly stated to the contrary, we suppose  $\lambda$  to have none of the values 0, 1,  $\infty$ . It follows that no two of the points  $z_1, z_2, z_3, z_4$  coincide unless three of them coincide.

Except in the trivial case that  $C_1$ ,  $C_2$ ,  $C_3$  are points,  $C_4$  is evidently a two-dimensional continuum and is not necessarily the entire plane. The envelope  $C_4$  is connected, for to join any pair of points  $z'_4, z''_4$  in  $C_4$  by a curve in  $C_4$ , we need merely to choose any set of points corresponding to each,  $z'_1, z'_2, z'_3; z''_1, z''_2, z''_3$ , in the proper regions. Join  $z'_1$  and  $z''_1$  by a continuous curve which lies in  $C_1$ , and similarly join  $z'_2$  and  $z''_2$ , and  $z'_3$  and  $z''_3$ , by continuous curves in  $C_2$  and  $C_3$  respectively. Allow  $z_1, z_2, z_3$  to move from  $z'_1, z'_2, z'_3$  to  $z''_1, z''_2, z''_3$  along these respective curves. The point  $z_4$  corresponding moves from  $z'_4$  to  $z''_4$  in  $C_4$  and along a curve which is continuous because  $z_4$  is a linear function of  $z_1, z_2, z_3$ .

Our next remark is stated explicitly as a lemma. It is readily stated and established for regions whose boundaries are curves much more general than circles, but we consider here merely the form under the hypothesis of Theorem II and for application to the proof of that theorem.

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advantages over the following suggested method of proof. The theorem is evidently true when  $C_1$ ,  $C_2$ , and  $C_3$  are points. The theorem is easily proved when  $C_1$  and  $C_2$  are points but  $C_3$  is not a point. By taking the envelope of the circular region  $C_4$  in the preceding degenerate case, the theorem can be proved when  $C_1$  is a point but neither  $C_2$  nor  $C_3$  is a point. The envelope of the region  $C_4$  in this last degenerate case, as  $z_1$  is allowed to vary over a region  $C_1$  not a point, gives the envelope of  $z_4$  for the theorem in its generality. I have not been able to carry through the actual analytic determination of the envelope by this method because the algebraic work is too laborious.

This suggested method of proof, however, shows at once that the boundary of the region  $C_4$  in the general case is an algebraic curve or at least part of an algebraic curve.

It seems to me likely that Theorem II is true also when  $\lambda$  is imaginary, but I have not carried through the proof in detail.

In general the relation of the regions  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  is not reciprocal. For example if  $C_1$  is a point but neither  $C_2$  nor  $C_3$  is a point and if these regions lead to the fourth region  $C_4$ , then if we choose the circular regions  $C_2$ ,  $C_3$ ,  $C_4$  as the original circular regions of the lemma, we cannot for any choice of  $\lambda$  be led to the region  $C_1$ . This lack of reciprocity does not depend on the degeneracy of one of the regions  $C_1, C_2, C_3, C_4$ .

**LEMMA III.** *If the point  $z_4$  is on but not at a vertex of the boundary of  $C_4$ ,\* then any set of points  $z_1, z_2, z_3$  corresponding lie on the boundaries of the respective regions  $C_1, C_2, C_3$ ; the circle  $C$  through the points  $z_1, z_2, z_3, z_4$  cuts the circles  $C_1, C_2, C_3$  all at angles of the same magnitude; and if  $C$  is transformed into a straight line, the lines tangent to the circles  $C_1, C_2$ , and  $C_3$  at the points  $z_1, z_2, z_3$  respectively are parallel.*

The following proof is formulated only for the general case that none of the circles  $C_1, C_2, C_3$  is a null circle, but no essential modification is necessary to include the degenerate cases.

When  $z_2$  and  $z_3$ , and also the circle  $C$  are kept fixed, a continuous motion of  $z_1$  along  $C$  also causes  $z_4$  to move continuously along  $C$ . If the direction of motion of  $z_1$  is reversed, the direction of motion of  $z_4$  is also reversed. Hence  $z_4$  is not on the boundary of  $C_4$  unless  $z_1$  is on the boundary of  $C_1$ , and as can be shown in an analogous manner, not unless  $z_2$  and  $z_3$  are on the boundaries of  $C_2$  and  $C_3$  respectively. The region  $C_4$  is closed since the regions  $C_1, C_2$ , and  $C_3$  are closed.

Let  $P$  be any point of the boundary of  $C_4$ . Transform  $P$  to infinity, so that the corresponding points  $z_1, z_2, z_3$  lie on the same line  $L$ . We assume at first that  $L$  is not tangent to any of the circles  $C_1, C_2, C_3$  nor to the boundary of  $C_4$ . The relative positions of the points  $z_1, z_2, z_3$  on  $L$  together with the sense along  $L$  in which the region  $C_1$  extends from  $z_1$  determine uniquely the sense along  $L$  in which the regions  $C_2, C_3, C_4$  must extend from  $z_2, z_3, P$  respectively. There is evidently a segment of  $L$  terminated by  $P$  composed entirely of points in  $C_4$ . If the lines tangent to the circles  $C_2$  and  $C_3$  at the points  $z_2$  and  $z_3$  are not parallel, it is possible slightly to rotate  $L$  about  $z_1$  in one direction or the other into a new position  $L'$  and to determine a point  $z_2''$  on  $L'$  and on the circle  $C_2$  and a point  $z_3''$  on  $L'$  and *interior to the region  $C_3$*  such that the triangles  $z_1 z_2 z_2''$  and  $z_1 z_3 z_3''$  are similar and hence we have the relation

$$(z_1, z_2'', z_3'', P) = \lambda.$$

Then  $z_3''$  can be moved in either sense along the line  $L'$  and still remain in its proper envelope, so there are corresponding points  $z_4''$  on  $L'$  in either sense from  $P$ . Moreover, this is true for every position of  $L'$  if the angle from  $L$  to  $L'$  is in the proper sense and is sufficiently small, so if we transform  $P$  to the finite part of the plane and  $z_1$  to infinity and notice that the lines  $L'$  are lines through the point  $P$ , it becomes evident that there are points  $z_4$  in the neighborhood of  $P$  on any line  $L'$  through  $P$  which lies within a certain sector whose vertex is  $P$ , and there are points  $z_4$  on  $L'$  in both directions

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\* It is of course true that the boundary of  $C_4$  has no vertices, but that fact has not yet been proved.

from  $P$ . Hence if  $P$  is actually on the boundary of  $C_4$ , it must lie at a vertex of that boundary.\*

The proof thus far has been formulated to prove that when  $P$  is at infinity the lines tangent to the circles  $C_2$  and  $C_3$  at  $z_2$  and  $z_3$  are parallel. The notation of the proof can easily be modified to show that the lines tangent to the circles  $C_1$  and  $C_2$  at  $z_1$  and  $z_2$  are parallel, and hence the lines tangent to  $C_1$ ,  $C_2$ ,  $C_3$  at  $z_1$ ,  $z_2$ ,  $z_3$  are parallel.

This same method of reasoning is readily used to prove that if the circle  $C$  of the lemma is tangent to one or two of the circles  $C_1$ ,  $C_2$ ,  $C_3$  at the respective points  $z_1$ ,  $z_2$ ,  $z_3$  but is not tangent to all these circles, the boundary of  $C_4$  has a vertex at  $z_4$ . The circle  $C$  is not tangent to the boundary of  $C_4$  unless  $C$  is tangent to  $C_1$ ,  $C_2$ , and  $C_3$ . This consideration completes the proof of Lemma III.

It is desirable to make a revision in our use of the term *angle between two circles*. With Coolidge,† we consider circles to be described by a point moving in a counter-clockwise sense, and define the angle between two circles to be the angle between the half-tangents drawn at the intersection in the sense of description of the circles. When we are concerned with a single straight line, either sense may be given to it. We shall use this convention in proving the following lemma, which is a result purely of circle geometry which has not necessarily any connection with Theorem II. As stated and proved, it is slightly more general than is necessary for its application in the proof of that theorem.

**LEMMA IV.** *Suppose a variable circle  $C$  either to cut three distinct fixed non-coaxial circles  $C_1$ ,  $C_2$ ,  $C_3$  all at the same angle or to cut a definite one of those circles at an angle supplementary to the angle cut on the other two. If the points  $z_1$ ,  $z_2$ ,  $z_3$  are chosen as intersections of  $C$  with  $C_1$ ,  $C_2$ ,  $C_3$  respectively such that when  $C$  is transformed into a straight line the lines tangent to  $C_1$ ,  $C_2$ ,  $C_3$  at  $z_1$ ,  $z_2$ ,  $z_3$  are all parallel, then the locus of the point  $z_4$  defined by the real constant cross-ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

*is a circle  $C_4$  which is also cut by  $C$  at an angle equal or supplementary to the angles cut on  $C_1$ ,  $C_2$ ,  $C_3$ .‡*

This lemma is not true if the circles  $C_1$ ,  $C_2$ ,  $C_3$  are coaxial circles having no point in common. For transform these circles into concentric circles. Then

\* The method of proof used in this paragraph was suggested to me by Professor Birkhoff.

† *A treatise on the circle and the sphere*, p. 108.

‡ We remark that the circle  $C_4$  can be constructed by ruler and compass whenever  $\lambda$  is rational or in fact whenever  $\lambda$  is given geometrically. For the circle  $C$  can be constructed by ruler and compass in any position; cf. Coolidge, l. c., p. 173. Hence we can determine any number of sets of points  $z_1$ ,  $z_2$ ,  $z_3$  and therefore construct any number of points  $z_4$ , which enables us to construct  $C_4$ .

the circle  $C$  is a straight line orthogonal to these circles,  $C$  has two intersections with each, and on any particular circle  $C$  the points  $z_1, z_2, z_3$  may be chosen on their proper circles so as to lead to four circles of type  $C_4$ , in general distinct, and concentric with  $C_1, C_2, C_3$ . All these four circles of type  $C_4$  form the locus of points  $z_4$ . The situation is essentially the same if  $C_1, C_2, C_3$  are coaxial circles having two common points; we are led to four circles  $C_4$  which are in general distinct. But if we suppose  $C$  to vary continuously and also the points  $z_1, z_2, z_3, z_4$  each to vary in one sense continuously, although of course we allow these points to go to infinity but not to occupy any position more than once, the lemma is true even for coaxial circles having no point or two points in common. These situations are included in the detailed treatments given under Cases I and II below.

This lemma breaks down also if the circles  $C_1, C_2, C_3$  are coaxial circles all tangent at a single point, for we can consider the three points  $z_1, z_2, z_3$  to coincide at that point; any circle  $C$  through that point satisfies the conditions of the lemma, any point of  $C$  can be chosen as  $z_4$ , whence it appears that the locus of  $z_4$  is then the entire plane. But if we make not only our previous convention but in addition the convention that not all of the points  $z_1, z_2, z_3$  shall lie at a point common to the three circles unless the fourth point coincides with them, then the lemma remains true. This situation is treated in detail under Case IV below.

The lemma is true but trivial in the degenerate cases  $\lambda = 0, 1$ , or  $\infty$ , for in these cases  $z_4$  coincides with  $z_3, z_2$ , or  $z_1$  respectively. The case that  $C_1, C_2$ , and  $C_3$  are all null circles is likewise trivial. In the consideration of other cases we shall use the following theorem:

**THEOREM.** *If three circles be given not all tangent at one point, the circles cutting them at equal angles form a coaxial system, as do those cutting one at angles supplementary to the angles cut on the other two.\**

Then as the circle  $C$  of Lemma IV varies, it always belongs to a definite coaxial system, unless  $C_1, C_2, C_3$  are all tangent at a single point. This system may consist of (Case I) circles through two points, (Case II) non-intersecting circles, or (Case III) circles tangent to a line at a single point. Under Case IV will be treated the situation when  $C_1, C_2, C_3$  are all tangent at a point. We consider these cases in order.

In Case I, transform to infinity one of the two points through which the coaxial family  $C$  passes, so that this family becomes the straight lines through a finite point  $q$  of the plane. In general  $q$  will be a center of similitude for each pair of the circles  $C_1, C_2$ , and  $C_3$ . These circles may or may not surround  $q$ .

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\* This statement differs from that of Coolidge, l. c., p. 111, Theorem 219, for we have adjoined to the plane the point at infinity. Theorem 220 seems to be erroneous; compare the four circles  $C_1, C_2, C_3, C_4$  of Lemma IV.

Let  $z_4$  be any point corresponding to the points  $z_1, z_2, z_3$  on  $C_1, C_2, C_3$  respectively. These four points lie on the line  $qz_4$ , and we have supposed that the lines tangent to  $C_1, C_2, C_3$  at the points  $z_1, z_2, z_3$  are parallel. Then when the line  $qz_4$  (that is, the circle  $C$ ) rotates about  $q$ , it will be seen that the point  $z_4$  as determined by its constant cross-ratio with  $z_1, z_2, z_3$  will trace a circle  $C_4$  such that  $q$  is a center of similitude for any of the pairs of circles  $C_1, C_2, C_3, C_4$ . If these circles do not surround  $q$ , they have two common tangents belonging to the family  $C$ , and the properly chosen cross-ratio of the points of tangency is  $\lambda$ . If  $C_1, C_2$ , and  $C_3$  are coaxial,  $C_4$  is coaxial with them. Perhaps it is worth noticing that any circle  $C_4$  such that  $q$  is a center of similitude for any pair of the circles  $C_1, C_2, C_3, C_4$  is the circle  $C_4$  of the lemma for a proper choice of  $\lambda$ ; in particular  $C_4$  may be the point  $q$  or the point at infinity.

Under Case I there are some special situations to be included. If one or more of the circles  $C_1, C_2, C_3$  passes through  $q$ , then each of the other circles if not a null circle either is tangent to that circle at  $q$  or is a line parallel to the line tangent to that circle at  $q$ . If two of the original circles, for definiteness  $C_1$  and  $C_2$ , are tangent at  $q$  and the other circle  $C_3$  is a line parallel to their common tangent at  $q$ , then either  $z_4$  coincides with  $z_1$  and  $z_2$  at  $q$ , or  $z_3$  remains at infinity during the motion of  $C$  while  $z_4$  traces a circle coaxial with  $C_1$  and  $C_2$ ; in particular this circle  $C_4$  may be the null circle  $q$ . The four circles  $C_1, C_2, C_3, C_4$  have a common tangent circle, namely the line tangent to  $C_1, C_2, C_4$  at  $q$ . In the case just considered, one of the circles which passes through  $q$ , for definiteness  $C_1$ , may be tangent at  $q$  to the second circle  $C_2$  which is a straight line. The circle  $C_3$  is a line parallel to  $C_2$ . When the circle  $C$  varies,  $z_4$  coincides with  $z_1$  and  $z_2$  at  $q$ ,  $z_4$  coincides with  $z_2$  and  $z_3$  at infinity, or the circle  $C$  coincides with  $C_2$ ,  $z_1$  with  $q$ , and  $z_3$  with the point at infinity, while  $z_2$  traces the line  $C_2$  and hence  $z_4$  also traces  $C_2$ . The circles  $C_1, C_2, C_3, C_4$  have a common tangent circle  $C_2$ . If one of the original circles, for definiteness  $C_1$ , passes through  $q$  and the circles  $C_2$  and  $C_3$  are lines parallel to the tangent to  $C_1$  and  $q$ , then the circle  $C_4$  is a circle coaxial with  $C_2$  and  $C_3$  which may be the point at infinity. The four circles  $C_1, C_2, C_3, C_4$  have as common tangent circle the line tangent to  $C_1$  at  $q$ .

The general situation of Case I is not essentially changed and requires no further discussion if one of the circles  $C_1, C_2, C_3$  is a point ( $q$  or the point at infinity) or if two of them are points ( $q$  and the point at infinity), except when at least one of the null circles lies on one of the non-null circles. In particular, if two circles, for example  $C_1$  and  $C_2$ , are null circles and one of them (say  $C_2$ ) lies on the non-null circle  $C_3$ , the locus of  $z_4$  is a circle  $C_4$  tangent to the circle  $C_3$  at the point  $C_2$ . If the two null circles  $C_1$  and  $C_2$  both lie on the non-null circle  $C_3$ , the circle  $C$  is effectually the circle  $C_3$ , and  $C_4$  coincides with  $C_3$ .

The special situations which we have considered under Case I may similarly degenerate by having one of the original circles a null circle. We shall discuss merely some typical examples. If  $C_1$  and  $C_2$  are tangent at  $q$  and  $C_3$  is a null circle at infinity,  $C_4$  is a circle tangent to  $C_1$  and  $C_2$  at  $q$  and may be the point  $q$  itself. If  $C_1$  is a null circle at  $q$ , if  $C_2$  is a circle passing through  $q$ , and if  $C_3$  is a line parallel to the tangent to  $C_2$  at  $q$ ,  $C_4$  is a circle tangent to  $C_2$  at  $q$ . If  $C_1$  is a null circle at  $q$ , if  $C_2$  is a line passing through  $q$ , and  $C_3$  is a line parallel to  $C_2$ , then  $C$  is essentially the single circle  $C_2$ , and  $C_4$  coincides with  $C_2$ .

In Case II, the coaxial family  $C$  is composed of circles having no point in common, and hence there are two null circles of the family. Transform one of these null circles to infinity, so that the family  $C$  becomes a family of circles with a common center  $p$ . In the general case, the circles  $C_1$ ,  $C_2$ , and  $C_3$  are all of equal radii and any of them can be brought into coincidence with any other of them by a rotation about  $p$ . The point  $p$  is outside, on, or within all three circles according as it is outside, on, or within any one of them. Choose any point  $z_4$  of the lemma; then  $z_1, z_2, z_3, z_4$  lie on the circle  $C$  whose center is  $p$ . As  $C$  varies, its radius simply increases or decreases, and  $z_1, z_2, z_3$  rotate about  $p$  so that the angles  $z_2 p z_3, z_3 p z_1, z_1 p z_2$  remain constant. Hence  $z_4$  traces a circle  $C_4$  whose radius is equal to the common radius of  $C_1, C_2$ , and  $C_3$ ; moreover any two of the four circles  $C_1, C_2, C_3, C_4$  can be brought into coincidence by a rotation about  $p$ . The four circles have two common tangent circles which belong to the family  $C$ , one of which may be the point  $p$ . The properly chosen cross-ratio of the points of tangency of a tangent circle is  $\lambda$ . Any circle is the circle  $C_4$  of the lemma for a proper choice of  $\lambda$  provided it can be brought into coincidence with any of the circles  $C_1, C_2, C_3$  by a rotation about  $p$ .

Another situation that may arise under Case II is that  $C_1, C_2$ , and  $C_3$  are straight lines (that is, coaxial circles) through  $p$  and the point at infinity; then the locus of  $z_4$  is a circle  $C_4$  coaxial with them. There remains also the possibility that  $C_1, C_2, C_3$  are straight lines all at the same distance from  $p$ . Then the circle  $C_4$  is a line also at this same distance from  $p$ . There is a circle belonging to the family  $C$  which is tangent to  $C_1, C_2, C_3, C_4$ , and as before the cross-ratio of the points of contact is  $\lambda$ .

In Case III, the circles  $C$  belong to a coaxial family of circles all tangent at a point  $n$ , which point we transform to infinity. The circles  $C$  become parallel lines and in general  $C_1, C_2, C_3$  become equal circles whose centers are collinear. As  $C$  moves parallel to itself, the points  $z_1, z_2, z_3$  remain at equal distances from each other. The locus of  $z_4$  either is a circle  $C_4$  equal to  $C_1, C_2$ , and  $C_3$  whose center is collinear with their centers or is the point at infinity. The four circles have two common tangent circles which belong

to the family  $C$ , and the cross-ratio of the points of tangency of each of these circles is  $\lambda$ .

A degenerate case that should be mentioned is that the point  $n$  itself is one of the circles  $C_1, C_2, C_3$ . The results are essentially the same as in the general situation. In both the degenerate and the general situations any circle  $C_4$  equal to  $C_1, C_2, C_3$  and whose center is collinear with their centers is the circle  $C_4$  of the lemma if  $\lambda$  is properly chosen.

A special case also occurs if one of the original circles, for definiteness  $C_1$ , is a straight line and the other two circles are straight lines parallel to the reflection of  $C_1$  in any of the circles  $C$ . When  $C$  varies, either  $z_4$  coincides with  $z_2$  and  $z_3$  at infinity, or  $z_1$  is at infinity and  $z_4$  traces a line parallel to  $C_2$  and  $C_3$ .

A degenerate case occurs if one of the original circles, say  $C_3$ , is the point at infinity, while  $C_1$  and  $C_2$  are the reflections of each other in one of the circles  $C$ . Under the conditions of the lemma  $z_4$  must coincide with  $z_3$  at infinity, so  $C_4$  coincides with  $C_3$ .

In Case IV, the circles  $C_1, C_2, C_3$  are all tangent at a point  $m$ . Transform  $m$  to infinity, so that in any non-degenerate case  $C_1, C_2, C_3$  become parallel lines. Under our convention that not all of the points  $z_1, z_2, z_3$  shall lie at  $m$  unless  $z_4$  coincides with them, we are led to four circles (in general distinct) according as we allow any one of the points  $z_1, z_2, z_3$  or none of them constantly to lie at infinity. The additional convention already made that  $z_1, z_2, z_3, z_4$  shall vary continuously in one sense and never coincide with any previous position enables us to choose simply one of these circles. The circle  $C$  is any straight line, and  $z_4$  is either the intersection of  $C$  with a straight line  $C_4$  parallel to  $C_1, C_2, C_3$  or if none of the points  $z_1, z_2, z_3$  is at infinity,  $z_4$  may be constantly the point at infinity. The circles  $C_1, C_2, C_3, C_4$  are all tangent at  $m$ .

Under Case IV should be mentioned the degenerate case that one of the circles  $C_1, C_2, C_3$  is a null circle lying at the point of tangency of the other two circles. Our conventions enable us to choose a circle  $C_4$  coaxial with  $C_1, C_2, C_3$ .

The proof of Lemma IV is now complete. It will be noticed that except in the special and degenerate cases, the result is entirely symmetric with respect to the four circles  $C_1, C_2, C_3, C_4$ . If we commence by choosing any three of those four circles and choose  $\lambda$  properly we shall be led to the other circle. If the last clause in the statement of the lemma is omitted, the lemma is true even if  $\lambda$  is not real.

There is a lemma corresponding to Lemma IV if we suppose two of the original circles, for example  $C_1$  and  $C_2$ , to coincide, but suppose  $C_3$  not to coincide with them. If we leave aside the easily treated cases  $\lambda = 0, 1$ , or  $\infty$ ,

we find either that the points  $z_1$  and  $z_2$  coincide on  $C_1$ , in which case  $z_4$  coincides with them and traces the circle  $C_1$ , or that if  $C_1$  is a non-null circle  $z_1$  and  $z_2$  do not coincide. In the latter case we are supposing the tangents to  $C$  at  $z_1$  and  $z_2$  to be parallel if  $C$  is transformed into a straight line and hence  $C$  must be orthogonal to  $C_1$  and therefore by the conditions of the lemma also orthogonal to  $C_3$ . As before, when the circle  $C$  varies it constantly belongs to a definite coaxial system. The reader will easily treat the cases corresponding to Cases I, II, and III above, and also the degenerate case that  $C_3$  is a null circle lying on  $C_1$  and  $C_2$ . The results in the general case are quite analogous to the previous results if we notice that  $C_1$ ,  $C_2$ , and  $C_3$  are coaxial. For if  $C_3$  is not a null circle,  $C$  cuts  $C_3$  in two distinct points, and by their cross-ratio with  $z_1$  and  $z_2$  these lead to *two distinct circles*  $C_4$  in addition to the circle  $C_1$ . Both of these new circles  $C_4$  belong to the coaxial family determined by  $C_1$  and  $C_3$ ; as  $C$  moves it is constantly orthogonal to  $C_4$  as well as to  $C_1$ ,  $C_2$ ,  $C_3$ . In general, then, the locus of  $z_4$  when  $C_1$  and  $C_2$  coincide is  $C_1$  and two other circles of the coaxial family determined by  $C_1$  and  $C_3$ . These two other circles may in a degenerate case coincide, as the reader can easily determine. The convention formerly made, that the points  $z_1$ ,  $z_2$ ,  $z_3$ ,  $z_4$  vary in one sense continuously will of course restrict the locus of  $z_4$  simply to one circle.

When the three circles  $C_1$ ,  $C_2$ ,  $C_3$  coincide, we must consider  $C$  to coincide with them, or else at least two of the points  $z_1$ ,  $z_2$ ,  $z_3$  to coincide and hence  $z_4$  to coincide with them. That is, the circle  $C_4$  corresponding to the circle  $C_4$  of the lemma is the circle  $C_1$ .

Lemmas III and IV with the discussion supplementary to the latter do not give us immediately all the material necessary for the proof of Theorem II. For if  $C_1$ ,  $C_2$ ,  $C_3$  are coaxial there are four circles, not necessarily all distinct, of the type  $C_4$  of the lemma. If  $C_1$ ,  $C_2$ ,  $C_3$  are not coaxial there are also four circles, not necessarily all distinct, of the type  $C_4$  of the lemma, according as  $C$  cuts all the circles  $C_1$ ,  $C_2$ ,  $C_3$  at equal angles or cuts one at an angle supplementary to the angle cut on the other two. It is conceivable that the boundary of the region  $C_4$  of Theorem II should consist of arcs of more than one distinct circle; we proceed to show that this is in fact never the case.\* The following lemma is essential in our proof.

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\* Whether the boundary of the region  $C_4$  corresponds to motion of  $C$  cutting the three original circles at the same angle or a definite one of those circles at an angle supplementary to the angle cut on the other two depends on the relative positions of those circles, on whether the various regions are interior or exterior to their bounding circles, and on the value of  $\lambda$ —in short, on the order of the points  $z_1$ ,  $z_2$ ,  $z_3$ ,  $z_4$  on the circle  $C$ . When the regions  $C_1$ ,  $C_2$ ,  $C_3$  are mutually external it is easy to prove by reasoning similar to that used in the proof of Lemma III that an arc of only one of the circles of type  $C_4$  can be a part of the boundary of the region  $C_4$ . This fact can also be proved in the general case by that same method of reasoning, but the proof given in detail below is perhaps more satisfactory. It is desirable

**LEMMA V.** *In Theorem II, whenever the envelope of  $z_4$  is not the entire plane, there is a circle  $S$  orthogonal to the four circles  $C_1, C_2, C_3, C_4$ .*

Whenever the regions  $C_1, C_2, C_3$  have a common point, we may consider  $z_1, z_2, z_3$  to coincide at that point, and consider the cross-ratio of any point  $z_4$  in the plane with those three points to have the value  $\lambda$ , so the envelope of  $z_4$  is the entire plane. In any other case there is a circle  $S$  orthogonal to the circles  $C_1, C_2, C_3$ . If not every pair of these three original circles intersect, choose two of them which do not intersect, and there will be two points inverse respecting both circles (these points are the null circles of the coaxial family determined by the two circles). Take the inverse of one of those points in the third of the original circles and pass a new circle  $S$  through all three points. Then  $S$  is orthogonal to the three original circles. If each of the circles  $C_1, C_2, C_3$  has a point in common with the other two, we can transform two of the circles into straight lines (if one of the circles is a null circle the other two circles pass through that null circle and hence the region  $C_4$  is the entire plane). If these two lines are not parallel, the third circle cannot be a straight line nor can it surround the intersection of the other two lines. Hence there is a circle orthogonal to all three circles. If the two lines are parallel the third circle cannot be a straight line. Then there is a circle, in this case a straight line, orthogonal to all three circles. This completes the proof of Lemma V.

Let us transform into a straight line any particular circle  $S$  orthogonal to the three original circles and let us suppose not every point of  $S$  to be a point of the region  $C_4$ ; for definiteness assume the point at infinity not to belong to  $C_4$ . The positions which each of the three points  $z_1, z_2, z_3$  of Theorem II may occupy fill an entire segment of  $S$ , and hence the points  $z_4$  on  $S$  which correspond to points  $z_1, z_2, z_3$  on  $S$  fill an entire segment of  $S$ ; we denote this segment by  $\sigma$ . The terminal points of the segment  $\sigma$  are the intersections of  $S$  with one of the circles of type  $C_4$  of Lemma IV; we denote that circle by  $C'_4$  and the other three circles of that type by  $C''_4, C'''_4, C''''_4$ . The entire configuration is symmetric with respect to  $S$ , so the centers of all the circles  $C'_4, C''_4, C'''_4, C''''_4$  lie on  $S$ . Moreover,  $S$  belongs to all four types of circles  $C$  of Lemma IV, since it is orthogonal to  $C_1, C_2, C_3$ . Hence the intersections of all the circles  $C''_4, C'''_4, C''''_4$  are points  $z_4$  which correspond to points  $z_1, z_2, z_3$  lying on  $S$ , and hence all those intersections lie on the segment  $\sigma$ . Then of the circles  $C''_4, C'''_4, C''''_4$  each is interior to or coincident with  $C'_4$ .

Either the entire interior or the entire exterior of each of the circles  $C'_4, C''_4, C'''_4, C''''_4$  belongs to the region  $C_4$ . For the points  $z_4$  which correspond to that most of the material making up that proof should be given anyway, as a test whether the region  $C_4$  is the entire plane, as giving a ruler-and-compass construction for the circle  $C_4$ , and as describing more in detail the entire situation with which we are concerned.

points  $z_1, z_2, z_3$  in the proper regions and on the circle  $C$  of Lemma IV fill an entire arc of  $C$ , extending from one intersection of  $C$  with the circle  $C_4$  to the other intersection. The entire exterior of our circle  $C'_4$  does not belong to the region  $C_4$ , for the point at infinity does not belong to that region. Hence the entire interior of  $C'_4$  does belong to the region  $C_4$ . No point external to  $C'_4$  can be a point of the boundary of  $C_4$ , for none of the circles  $C''_4, C'''_4, C''''_4$  has a point exterior to  $C'_4$ . Hence the region  $C_4$  is the interior of  $C'_4$ , under our assumption that not every point of  $S$  belongs to the region  $C_4$ .

Let us notice that we can allow any or all of the circles  $C_1, C_2, C_3$  to move continuously so as to remain orthogonal to  $S$ , so as never to intersect any former position, and so as always to enlarge the regions  $C_1, C_2, C_3$ . Then the circle  $C'_4$  grows larger and larger, never intersecting its former position, until it becomes the point at infinity, in which case the region  $C_4$  is the entire plane. If the regions  $C_1, C_2, C_3$  are enlarged still further, the region  $C_4$  still remains the entire plane.

Whether or not we assume that not every point of  $S$  belongs to the region  $C_4$ , we can start with a situation in which not every point of  $S$  is a point of  $C_4$  and enlarge the regions  $C_1, C_2, C_3$  in the manner described so as to attain any situation desired in which the region  $C_4$  is not the entire plane. At every stage the region  $C_4$  is a circular region. This completes the proof of Theorem II. We have also obtained a test whether or not the region  $C_4$  is the entire plane. *A necessary and sufficient condition that the region  $C_4$  of Theorem II be the entire plane is that the point  $z_4$  may occupy any position on  $S$  and still correspond to points  $z_1, z_2, z_3$  in their proper envelopes and also on  $S$ .*

The preceding developments give a comparatively simple ruler-and-compass construction for the circle  $C_4$ , whenever  $\lambda$  is rational or is given geometrically. The circle  $S$  can be constructed by ruler and compass.\* The two points of intersection of  $S$  and  $C_4$  can be determined by means of their cross-ratio with properly chosen intersections of  $S$  and  $C_1, C_2, C_3$ . Since  $S$  and  $C_4$  are orthogonal,  $C_4$  can then be constructed.

We shall apply Theorem II in proving our principal theorem.

**THEOREM III.** *Let  $f_1$  and  $f_2$  be binary forms of degrees  $p_1$  and  $p_2$  respectively, and let the circular regions  $C_1, C_2, C_3$  be the respective envelopes of  $m$  roots of  $f_1$ , the remaining  $p_1 - m$  roots of  $f_1$ , and all the roots of  $f_2$ . Denote by  $C_4$  the circular region which is the envelope of points  $z_4$  such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

*when  $z_1, z_2, z_3$  have the respective envelopes  $C_1, C_2, C_3$ . Then the envelope of*

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\* Coolidge, l. c., p. 173.

the roots of the jacobian of  $f_1$  and  $f_2$  is the region  $C_4$ , together with the regions  $C_1$ ,  $C_2$ ,  $C_3$  except that among the latter the corresponding region is to be omitted if any of the numbers  $m$ ,  $p_1 - m$ ,  $p_2$  is unity. If a region  $C_i$  ( $i = 1, 2, 3, 4$ ) has no point in common with any other of those regions which is a part of the envelope of the roots of the jacobian, it contains of those roots precisely  $m - 1$ ,  $p_1 - m - 1$ ,  $p_2 - 1$ , or 1 according as  $i = 1, 2, 3$ , or 4.

We shall first show by the aid of Lemmas I and II and of Theorems I and II that no point not in  $C_1$ ,  $C_2$ ,  $C_3$ , or  $C_4$  can be a root of the jacobian. For if a point  $z_4$  is not in  $C_1$ ,  $C_2$ , or  $C_3$  and is a root of the jacobian, it is a position of equilibrium and not of pseudo-equilibrium. The force at  $z_4$  will not be changed if we replace the particles in each of the regions  $C_1$ ,  $C_2$ ,  $C_3$  by the same number of coincident particles at points  $z_1$ ,  $z_2$ ,  $z_3$  in the respective regions. Then  $z_4$  is a position of equilibrium in the new field of force and hence by Lemma II we have

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

and therefore  $z_4$  lies in  $C_4$ .

Any point in  $C_4$  can be a root of the jacobian, for we need merely find points  $z_1$ ,  $z_2$ ,  $z_3$  in the regions  $C_1$ ,  $C_2$ ,  $C_3$  such that

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m}$$

and allow all the roots of the ground forms in each of those regions to coincide at those points. Any point of a region  $C_1$ ,  $C_2$ ,  $C_3$  which is the envelope of more than one root of a ground form can be a position of pseudo-equilibrium and hence a root of the jacobian. If any of the regions  $C_1$ ,  $C_2$ ,  $C_3$  is the envelope of merely one root of a ground form, then no point in that region but not in any other of the regions  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  can be a position of equilibrium or of pseudo-equilibrium and hence no such point can be a root of the jacobian. If a point is common to two of the regions  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  it is a point of  $C_4$  and hence is a point of the envelope of the roots of the jacobian.

We have now proved the theorem except for its last sentence, to the demonstration of which we now proceed. When the roots of the ground forms in the regions  $C_1$ ,  $C_2$ ,  $C_3$  coincide, the regions  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  contain respectively the following numbers of roots of the jacobian:  $m - 1$ ,  $p_1 - m - 1$ ,  $p_2 - 1$ , 1. The roots of the jacobian vary continuously when the roots of the ground forms vary continuously; no root of the jacobian can enter or leave any of the regions  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  which has no point in common with any other of those regions which is a part of the envelope of the roots of the jacobian.

The proof of Theorem III is now complete.\* It applies to the sphere as well as the plane, since everything essential in the theorem is invariant under stereographic projection.

Instead of considering primarily the jacobian of two binary forms as heretofore, we may consider a rational function  $f(z)$ , introduce homogeneous coördinates, and compute the value of the derivative  $f'(z)$  in terms of  $J$ , the jacobian of the binary forms which are the numerator and denominator of  $f(z)$ . We find that *the roots of  $f'(z)$  are the roots of  $J$  and a double root at infinity, except that when one of these points is also a pole of  $f(z)$  it cannot be a root of  $f'(z)$ .*† Application of Theorem III gives a theorem analogous to Theorem III, but which we state in a form slightly different from the statement of that theorem.

**THEOREM.** *If the circular regions  $C_1, C_2, C_3$  contain respectively  $m$  roots (or poles) of a rational function  $f(z)$  of degree  $p$ , all the remaining roots (or poles) of  $f(z)$ , and all the poles (or roots) of  $f(z)$ , then all the roots of  $f'(z)$  lie in the regions  $C_1, C_2, C_3$ , and a fourth circular region  $C_4$  determined as the envelope of points  $z_4$  such that*

$$(z_1, z_2, z_3, z_4) = \frac{p}{m},$$

*while the envelopes of  $z_1, z_2, z_3$  are respectively  $C_1, C_2, C_3$ ,—except that there are two roots at infinity if  $f(z)$  has no pole there. Except for these two additional roots, if any of the regions  $C_i$  ( $i = 1, 2, 3, 4$ ) has no point in common with any other of those regions which contains a root of  $f'(z)$ , then that region contains the following number of roots of  $f'(z)$  for  $i = 1, 2, 3, 4$  respectively:*

$$m - 1, \quad p - m - 1, \quad q_3 - 1, \quad 1;$$

or

$$q_1 - 1, \quad q_2 - 1, \quad p - 1, \quad 1,$$

*according as  $C_1$  contains  $m$  roots or  $m$  poles of  $f(z)$ ; here  $q_i$  indicates the number of distinct poles of  $f(z)$  in  $C_i$ .*

Perhaps the following special cases of this theorem are worth stating explicitly.

*If  $f(z)$  is a rational function whose  $m_1$  finite roots (or poles) lie on or within a circle  $C_1$  with center  $\alpha_1$  and radius  $r_1$  and whose  $m_2$  finite poles (or roots) lie on or within a circle  $C_2$  with center  $\alpha_2$  and radius  $r_2$ , and if  $m_1 > m_2 > 0$ , then*

\* It may be noticed that this proof does not explicitly use the fact that  $C_4$  is a circular region.

If  $C_1, C_2, C_3$  are coaxial circles with no point in common, Theorem III reduces essentially to Theorem II (I, p. 294). If  $m = 0$  or  $p_1 - m = 0$ , the regions  $C_1, C_2$ , and  $C_4$  can be considered to coincide; this gives Theorem III (I, p. 296), which is due to Bôcher.

† See I, p. 297.

all the finite roots of  $f'(z)$  lie in  $C_1$ ,  $C_2$ , and a third circle  $C_3$  whose center is

$$\frac{m_1 \alpha_2 - m_2 \alpha_1}{m_1 - m_2}$$

and radius

$$\frac{m_1 r_2 + m_2 r_1}{m_1 - m_2}.$$

If  $f(z)$  has no finite multiple poles, and if  $C_1$ ,  $C_2$ ,  $C_3$  are mutually external, they contain respectively the following numbers of roots of  $f'(z)$ :  $m_1 - 1$ ,  $m_2 - 1$ , 1. Under the given hypothesis, if  $m_1 = m_2$  and if  $C_1$  and  $C_2$  are mutually external, these circles contain all the finite roots of  $f'(z)$ .\*

If  $f(z)$  is a polynomial  $m_1$  of whose roots lie on or within a circle  $C_1$  whose center is  $\alpha_1$  and radius  $r_1$ , and if the remaining  $m_2$  roots lie on or within a circle  $C_2$  whose center is  $\alpha_2$  and radius  $r_2$ , then all the roots of  $f'(z)$  lie on or within  $C_1$ ,  $C_2$ , and a third circle  $C_3$  whose center is

$$\frac{m_1 \alpha_2 + m_2 \alpha_1}{m_1 + m_2}$$

and radius

$$\frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}.$$

If these circles are mutually external, they contain respectively the following number of roots of  $f'(z)$ :  $m_1 - 1$ ,  $m_2 - 1$ , 1.

If  $f(z)$  is a polynomial of degree  $n$  with a  $k$ -fold root at  $P$ , and with the remaining  $n - k$  roots in a circular region  $C$ , then all the roots of  $f'(z)$  lie at  $P$ , in  $C$ , and in a circular region  $C'$  obtained by shrinking  $C$  toward  $P$  as center of similitude in the ratio  $1 : k/n$ . If  $C$  and  $C'$  have no point in common they contain respectively  $n - k - 1$  roots and 1 root of  $f'(z)$ .†

A special case of this last theorem is the following

**THEOREM.** If a circle includes all the roots of a polynomial  $f(z)$ , it also includes all the roots of  $f'(z)$ .

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\* A more restricted theorem than this has been proved not merely for rational functions but also for the quotient of two entire functions. See M. B. Porter, Proceedings of the National Academy of Sciences, vol. 2 (1916), pp. 247, 335.

There is no theorem analogous to the theorem of the present paper if  $m_1 = m_2$  and if  $C_1$  and  $C_2$  are not mutually external. For we may consider all the roots and all the poles of  $f(z)$  to coincide, so that  $f(z)$  reduces to a constant and every point of the plane is a root of  $f'(z)$ .

† This theorem is true whether the circle  $C$  surrounds, passes through, or does not surround  $P$ , and whether the region  $C$  is interior or exterior to the circle  $C$ . The special case where  $P$  is the center of the circle  $C$  and the region  $C$  is external to that circle was pointed out in a footnote, I, p. 298. The special case where  $C$  does not surround  $P$  and the region  $C$  is interior to the circle  $C$  was pointed out to me by Professor D. R. Curtiss.

The latter theorem is equivalent to the well-known theorem of Lucas:

*If all the roots of a polynomial  $f(z)$  lie on or within any convex polygon, then all the roots of  $f'(z)$  lie on or within that polygon.*

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May, 1920.

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